

On Holomorphic Artin L-functions

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Abstract

Let K/\mathbb{Q} be a finite Galois extension, $s_0 \in \mathbb{C} \setminus \{1\}$, $Hol(s_0)$ the semigroup of Artin L-functions holomorphic at s_0 . If the Galois group is almost monomial then Artin's L-functions are holomorphic at s_0 if and only if $Hol(s_0)$ is factorial. This holds also if s_0 is a zero of an irreducible L-function of dimension ≤ 2 , without any condition on the Galois group.

Key words: Artin L-function; Artin's holomorphy conjecture
MSC: 11R42

1. Introduction

Let K/\mathbb{Q} be a finite Galois extension with the Galois group G , χ_1, \dots, χ_r the irreducible characters of G with the dimensions $d_1 := \chi_1(1), \dots, d_r := \chi_r(1)$, $f_1 = L(s, \chi_1, K/\mathbb{Q}), \dots, f_r = L(s, \chi_r, K/\mathbb{Q})$ the corresponding Artin L-functions,

$$Ar := \{f_1^{k_1} \cdot \dots \cdot f_r^{k_r} \mid k_1 \geq 0, \dots, k_r \geq 0\}$$

the multiplicative semigroup of all L-functions. Artin proved that f_1, \dots, f_r are multiplicatively independent ([1], Satz 5, P. 106), so Ar is factorial with the set of primes $\{f_1, \dots, f_r\}$. For $s_0 \in \mathbb{C}, s_0 \neq 1$, let $Hol(s_0)$ be the

subsemigroup of Ar consisting of the L-functions which are holomorphic at s_0 . Artin conjectures that every L-function is holomorphic at s_0 . This is true for monomial Galois groups. If G is isomorphic to A_5 , the alternating group on five elements, it was proved in ([3], Theorem 3) that the L-functions are holomorphic at s_0 if and only if $Hol(s_0)$ is factorial. A finite group G is called *almost monomial* if for every distinct irreducible characters χ and ψ of G there exist a subgroup H of G and a linear character ψ of H such that the induced character ψ^G contains χ and does not contain ψ . Every monomial group and every quasi monomial group in the sense of ([2]) are almost monomial. The group A_5 is almost monomial.

Theorem 1. *If the Galois group is almost monomial, then the following assertions are equivalent:*

- 1) *Artin's conjecture is true: $Hol(s_0) = Ar$.*
- 2) *The semigroup $Hol(s_0)$ is factorial.*

Proof. 1) \Rightarrow 2): If the L-functions are holomorphic at s_0 , then $Hol(s_0) = Ar$ is factorial.

2) \Rightarrow 1): Suppose that Artin's conjecture is not true. Then there exists $1 \leq k \leq r$ such that

$$\text{ord}(f_k) < 0, \tag{1}$$

where $\text{ord}(f)$ denotes the order of the Artin L-function f at s_0 . The Dedekind zeta function ζ_K of K decomposes as

$$\zeta_K = f_1^{d_1} \cdot \dots \cdot f_r^{d_r}. \tag{2}$$

Since ζ_K is holomorphic in $\mathbb{C} \setminus \{1\}$ it holds that

$$\text{ord}(\zeta_K) \geq 0. \tag{3}$$

From (1), (2) and (3) it follows that there exists $l \in \{1, \dots, r\}$ such that

$$\text{ord}(f_l) > 0.$$

For $j \in \{1, \dots, r\}$ let

$$m_j := \min\{m \geq 0 : \text{ord}(f_l^m \cdot f_j) \geq 0\}.$$

Since the L-functions f_1, \dots, f_r are multiplicatively independent the elements $f_l^{m_1} \cdot f_1, \dots, f_l^{m_r} \cdot f_r$ are irreducible in $Hol(s_0)$. We have seen in [3], p. 2862, that $Hol(s_0)$ is a positive affine semigroup which generates the free abelian group $\{f_1^{k_1} \cdot \dots \cdot f_r^{k_r} \mid k_1 \in \mathbb{Z}_0, \dots, k_r \in \mathbb{Z}\}$ with the basis f_1, \dots, f_r . The Hilbert basis \mathcal{H} of $Hol(s_0)$ is the uniquely determined minimal system of generators

of $Hol(s_0)$, hence $Hol(s_0)$ is factorial if and only if \mathcal{H} has r elements. It follows that

$$\mathcal{H} = \{f_l^{m_1} \cdot f_1, \dots, f_l^{m_r} \cdot f_r\}.$$

From (1) it follows that $m_k > 0$. Since the Galois group G is almost monomial there exist a subgroup H of G and a linear character ψ of H such that the induced character ψ^G contains χ_k and does not contain χ_l . By classfield theory the Artin L-function $L(s, \psi^G, K/\mathbb{Q})$ is a Hecke L-function so it is holomorphic at s_0 . Then $L(s, \psi^G, K/\mathbb{Q})$ is a product of elements of \mathcal{H} . Since ψ^G contains χ_k the Artin L-function $L(s, \psi^G, K/\mathbb{Q})$ contains $L(s, \chi_k, K/\mathbb{Q}) = f_k$ so it contains $f_l^{m_k} \cdot f_k$ and so f_l since $m_k > 0$. On the other hand, since ψ^G does not contain χ_l the Artin L-function $L(s, \psi^G, K/\mathbb{Q})$ does not contain $L(s, \chi_l, K/\mathbb{Q}) = f_l$, a contradiction. □

We don't know whether any finite group G is almost monomial. We think that theorem 1 is true without no condition on the Galois group. We can prove only a partial result:

Theorem 2. *If s_0 is a zero of some f_l with $d_l \leq 2$ then the following assertions are equivalent:*

- 1) *Artin's conjecture is true: $Hol(s_0) = Ar$.*
- 2) *The semigroup $Hol(s_0)$ is factorial.*

Proof. 1) \Rightarrow 2) is clear.

2) \Rightarrow 1): Suppose that Artin's conjecture is not true. For $j \in \{1, \dots, r\}$ let

$$m_j := \min\{m \geq 0 : \text{ord}(f_l^m \cdot f_j) \geq 0\}.$$

As in the proof of theorem 1 we have that the Hilbert basis of $Hol(s_0)$ is

$$\mathcal{H} = \{f_l^{m_1} \cdot f_1, \dots, f_l^{m_r} \cdot f_r\}.$$

Since $\zeta_K \in Hol(s_0)$ there exist $a_1 \geq 0, \dots, a_r \geq 0$ such that

$$\zeta_K = \prod_{j=1}^r (f_l^{m_j} \cdot f_j)^{a_j},$$

so

$$f_1^{d_1} \cdot \dots \cdot f_r^{d_r} = \prod_{j=1}^r (f_l^{m_j} \cdot f_j)^{a_j},$$

$$d_l = a_l + \sum_{j=1, j \neq l}^r m_j a_j,$$

$$d_j = a_j, j \neq l$$

since f_1, \dots, f_r are multiplicatively independent and $m_l = 0$. Hence

$$d_l = a_l + \sum_{j=1, j \neq l}^r m_j d_j. \quad (4)$$

Suppose that there exists $k \neq l$ such that

$$\text{ord}(f_k) > 0.$$

For $j \in \{1, \dots, r\}$ let

$$n_j := \min\{m \geq 0 : \text{ord}(f_k^m \cdot f_j) \geq 0\}.$$

It follows that

$$\mathcal{H} = \{f_k^{n_1} \cdot f_1, \dots, f_k^{n_r} \cdot f_r\}$$

hence

$$\begin{aligned} \{f_l^{m_1} \cdot f_1, \dots, f_l^{m_r} \cdot f_r\} &= \{f_k^{n_1} \cdot f_1, \dots, f_k^{n_r} \cdot f_r\}, \\ f_l^{m_j} \cdot f_j &= f_k^{n_j} \cdot f_j \end{aligned}$$

for $j \neq k, l$,

$$f_l^{m_j} = f_k^{n_j},$$

$$m_j = n_j = 0,$$

$$\mathcal{H} = \{f_j : j \neq k, l\} \cup \{f_l^{m_k} \cdot f_k, f_l\} = \{f_j : j \neq k, l\} \cup \{f_k^{n_l} \cdot f_l, f_k\},$$

$$f_k = f_l^{m_k} \cdot f_k,$$

$$f_l^{m_k} = 1,$$

$$m_k = 0,$$

$$n_l = 0,$$

$$\mathcal{H} = \{f_1, \dots, f_r\}.$$

This means that f_1, \dots, f_r are holomorphic at s_0 , so Artin's conjecture is true, a contradiction. It follows that

$$\text{ord}(f_k) \leq 0 \text{ for every } k \neq l. \quad (5)$$

Since we have supposed that Artin's conjecture is not true there exists $k \neq l$ such that

$$\text{ord}(f_k) < 0.$$

Since $m_j = 0$ if $\text{ord}(f_j) = 0$ from (4) and (5) it follows that

$$d_l = a_l + \sum_{j:\text{ord}(f_j)<0} m_j d_j,$$

hence

$$d_l \geq \sum_{j:\text{ord}(f_j)<0} m_j d_j, \quad (6)$$

since $a_l \geq 0$. For any j with $\text{ord}(f_j) < 0$ we have that $d_j \geq 2$, since if $d_j = 1$ then by classfield theory the L-function f_j is a Hecke L-function so it is holomorphic at s_0 . Since $d_l \leq 2$ it follows from (6) that there exists only one k such that $\text{ord}(f_k) < 0$. This implies $d_k = 2$, $m_k = 1$, $d_l = 2$, $\mathcal{H} = \{f_j : j \neq k\} \cup \{f_l \cdot f_k\}$. By a result of Rhoades improving the theorem of Aramata-Brauer ([4], Theorem 2, p. 359) the function $\zeta_K \cdot f_k$ is holomorphic in $\mathbb{C} \setminus \{1\}$, so $\zeta_K \cdot f_k \in \text{Hol}(s_0)$. Then $\zeta_K \cdot f_k$ is a product of elements of \mathcal{H} : there exist $b_1, \dots, b_r \geq 0$ such that

$$\zeta_K \cdot f_k = \left(\prod_{j \neq k} f_j^{b_j} \right) \cdot (f_l \cdot f_k)^{b_k},$$

$$\left(\prod_{j \neq k} f_j^{d_j} \right) \cdot f_k^{d_k+1} = \left(\prod_{j \neq k} f_j^{b_j} \right) \cdot (f_l \cdot f_k)^{b_k},$$

$$b_k = d_k + 1, d_l = b_l + b_k,$$

$$d_l \geq d_k + 1 = 3,$$

in contradiction with $d_l = 2$.

□

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